MATH 208 — MIDTERM 1 — Autumn 2022

Professor Rekha Thomas

NAME (Please print): ______________________________

There are 4 problems. Show all of your work and justify your answers.

Problem	Score
1	
2	
3	
4	
Total	

- **All phones and headphones must be put away in your bag.**
- **No talking or looking around during the exam.** Any form of cheating will result in a zero on this exam.
- You are allowed one 8.5x11 sheet of notes (written on both sides) and a Texas Instruments TI-30X IIS calculator.

(1) Let S denote the set of solutions to the following system of linear equations:

$$\begin{array}{rcrcrcrcl} 2x_1 & + & x_2 & - & x_3 & + & x_4 & = & 0 \\ & - & x_2 & + & 2x_3 & + & x_4 & = & 0 \end{array}$$

(a) S is the intersection of planes.

(i) How many planes?

2 one corresponding to each equation

(ii) Where do the planes live?

in $\mathbb{R}^4$, each eq$^{\underline{n}}$ has 4 variables

(b) What do you expect S to look like? Why?

A 2-dim$^{\underline{l}}$ plane. in $\mathbb{R}^4$

Each eq$^{\underline{n}}$ gives a 3-d plane and each equation cuts dimension by 1. So you go from $\mathbb{R}^4 \longrightarrow$ 3-d plane $\longrightarrow$ 2d plane

(c) Compute S.

System in echelon form

$x_2 = 2x_3 + x_4$

$2x_1 = -x_2 + x_3 - x_4 = -2x_3 - x_4 + x_3 - x_4$

$= -x_3 - 2x_4$

$x_1 = -\frac{1}{2}x_3 - x_4$

$$S = \left\{ \begin{pmatrix} -\frac{1}{2}s - t \\ 2s + t \\ s \\ t \end{pmatrix} : s, t \in \mathbb{R} \right\}$$

(d) Using (c) find a set of vectors that span S. Explain what you did.

$$S = \left\{ \begin{pmatrix} -\frac{1}{2}s - t \\ 2s+t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} -\frac{1}{2} \\ 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} : s, t \in \mathbb{R} \right\}$$

$\therefore$ S is the span of $\left\{ \begin{pmatrix} -\frac{1}{2} \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

(e) How would you construct a new equation which can be added to the system so that the set of solutions does not change? Describe your plan in words and then write down such an equation.

Take a linear combination of the given equs.

<u>Ex</u> Adding the 2 eq$^{\underline{n}}$s we get

$$2x_1 + x_3 + 2x_4 = 0$$

(f) Using (c) can you write $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ in which every vector participates? Explain your strategy.

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the last col of given system and the others are the 1^{st}, 2^{nd}, 3^{rd} cols.

To write $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as a linear comb of other cols,

set $t = -1$, $s = 1$ Then sol$^{\underline{n}}$ is

$$\begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \\ -1 \end{pmatrix} \quad \therefore \quad \frac{1}{2}\begin{pmatrix} 2 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 1\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So, yes can write $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as a linear comb$^{\underline{n}}$ of rest w/ all vectors participating

(2) (a) What is the span of the following vectors? Use Gaussian elimination to answer this question. Show all your work and box your answer.

$$\underset{u_1}{\begin{pmatrix}1\\0\\-1\end{pmatrix}}, \underset{u_2}{\begin{pmatrix}2\\-1\\5\end{pmatrix}}, \underset{u_3}{\begin{pmatrix}0\\1\\-7\end{pmatrix}}, \underset{u_4}{\begin{pmatrix}1\\2\\-15\end{pmatrix}}$$

Pick $b = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} \in \mathbb{R}^3$ and try to solve $u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = b$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & b_1 \\ 0 & -1 & 1 & 2 & b_2 \\ -1 & 5 & -7 & -15 & b_3 \end{array}\right] \xrightarrow[R_3+R_1]{R_3 \leftarrow} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & b_1 \\ 0 & -1 & 1 & 2 & b_2 \\ 0 & 7 & -7 & -14 & b_1+b_3 \end{array}\right]$$

$$\xrightarrow[R_3+7R_2]{R_3 \leftarrow} \underbrace{\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & b_1 \\ 0 & -1 & 1 & 2 & b_2 \\ 0 & 0 & 0 & 0 & b_1+7b_2+b_3 \end{array}\right]}_{\text{system has a sol}^{\underline{n}}} \Rightarrow \text{Span of vectors} = \underbrace{\left\{ \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} \in \mathbb{R}^3 : b_1+7b_2+b_3=0 \right\}}_{\text{2 dim}^{\underline{l}} \text{ plane in } \mathbb{R}^3}$$

$\Longleftrightarrow b_1 + 7b_2 + b_3 = 0$

(b) Looking at the echelon form you computed in (a) find as many linearly independent vectors as you can among the original four vectors. Explain how you used the echelon form to arrive at your conclusion.

<u>Two</u> eg $\begin{pmatrix}1\\0\\-1\end{pmatrix}$ & $\begin{pmatrix}2\\-1\\5\end{pmatrix}$ since there are pivots in cols 1 & 2 of echelon form

i.e If we solved $\begin{pmatrix}1\\0\\-1\end{pmatrix}x_1 + \begin{pmatrix}2\\-1\\5\end{pmatrix}x_2 = \begin{pmatrix}0\\0\\0\end{pmatrix}$, the echelon form would be

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$ which says the system has only $x_1=0$ $x_2=0$ as sol$^{\underline{n}}$.

(3) In each case below find the values of t (when possible) for which the given vectors are linearly **dependent**. Give reasons in each case.

(a) $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}, \begin{pmatrix}5\\6\end{pmatrix}, \begin{pmatrix}t\\7\end{pmatrix}$

$\boxed{\text{Every } \cancel{\text{No}} \text{ value of } t}$ makes these vectors LD since 4 vectors in $\mathbb{R}^2$ are dependent always

(b) $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\t\\5\end{pmatrix}$

$\boxed{\text{No value of } t}$ works since LD $\Rightarrow$ $5 = 4 \cdot 3$ which is false

(c) $\begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}-1\\-1\\1\end{pmatrix}, \begin{pmatrix}1-t\\1-t\\-1+t\end{pmatrix}$

$\boxed{\text{All values of } t}$ make these vectors dependent

Since $\begin{pmatrix}1-t\\1-t\\-1+t\end{pmatrix} = \begin{pmatrix}1\\1\\-1\end{pmatrix} + t\begin{pmatrix}-1\\-1\\1\end{pmatrix}$ is a linear comb$^{\underline{n}}$ of the 1st two for all t

(d) $\underset{u_1}{\begin{pmatrix}1\\2\\3\\4\end{pmatrix}}, \underset{u_2}{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}}, \underset{b}{\begin{pmatrix}t\\t\\2\\2\end{pmatrix}}$

Vectors are dependent when $\boxed{t = 1}$

Solve $u_1 x + u_2 y = b$

$x + y = t$ $\quad$ $y = t/2$

$2x = t \Rightarrow x = t/2$

$3x + y = 2 \Rightarrow 2t = 2 \Rightarrow t = 1$

$4x = 2 \Rightarrow 2t = 2$ (same as before)

(4) Jake wants to find a polynomial function that passes through the points $(1,0),(2,0),(3,0)$ by solving a system of linear equations. In each case below,

(i) set up the equations he would solve,

(ii) decide whether he will be able to find such a polynomial, and

(iii) if so, how many.

No need to calculate the polynomial(s). In each question, mark the parts (i),(ii),(iii) clearly and give reasons for (ii) and (iii).

(a) A line $y = a_0 + a_1 x$.

(i)
$$0 = a_0 + a_1$$
$$0 = a_0 + 2a_1$$
$$0 = a_0 + 3a_1$$

(ii) $\left[\begin{array}{c|cc} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{array}\right] \to \left[\begin{array}{c|cc} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{array}\right] \to \left[\begin{array}{c|cc} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$

only sol$^{\underline{n}}$ is $a_0 = 0$, $a_1 = 0$ so YES!

(iii) So $y = 0$ is the only possibility

(b) A quadratic $y = a_0 + a_1 x + a_2 x^2$.

(i)
$$0 = a_0 + a_1 + a_2$$
$$0 = a_0 + 2a_1 + 4a_2$$
$$0 = a_0 + 3a_1 + 9a_2$$

solve

(ii) $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 0 \\ 1 & 3 & 9 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 5 & 0 \end{array}\right]$

$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$ Only sol$^{\underline{n}}$ is $a_0 = 0 = a_1 = a_2$

(iii) Gets $y = 0 = 0 + 0 \cdot x + 0 \cdot x^2$

1 sol$^{\underline{n}}$

(c) A cubic $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

(i) $0 = a_0 + a_1 + a_2 + a_3$

$0 = a_0 + 2a_1 + 4a_2 + 8a_3$

$0 = a_0 + 3a_1 + 9a_2 + 27a_3$

(ii) 3 eq$^{\underline{n}}$s in 4 variables, 0 is already a sol$^{\underline{n}}$

so system is consistent and has

(iii) infinitely many sol$^{\underline{n}}$s

YES, he will find infinitely many cubic polynomials that go through the 3 pts.